A simple model of the vertical–horizontal illusion
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1. Introduction

In the vertical–horizontal illusion, a vertical line appears longer than a horizontal one of the same physical length. In spite of its long history, a complete explanation of this phenomenon is still elusive (for a recent review, see Wolfe, Maloney, & Tam, 2005). One of the reasons for the elusiveness is that there are at least two separate factors at play (Künnapas, 1955). The first factor is a genuine anisotropy between vertical and horizontal segments, i.e. a bias to overestimate the vertical length. The second factor is a length bisection bias. According to this latter bias, a line that is bisected in two parts will appear shorter than if it were not interrupted (Finger & Spelt, 1947). The bisection bias is present in the ‘T’ configuration of the vertical–horizontal illusion where the horizontal segment is bisected by a dividing vertical line. This figure systematically leads to a stronger effect than the ‘L’-configuration where the two lines meet at their extremities because in the ‘T’, the bisection bias goes in the same direction as the anisotropy bias.

The purpose of the present work is not to explain the origin of either the anisotropy or the bisection bias. Instead, we are interested in providing an explicit quantitative model of the most basic vertical–horizontal figures to separate the relevant factors that contribute to this illusion. Once these factors are isolated, one can confidently focus on one or the other factors and search for an explanation.

Even though it is now well accepted that both the anisotropy and the bisection biases play a role in the vertical–horizontal illusion (e.g. Avery & Day, 1969), three questions remain unanswered. First, it is not yet clear whether these two factors are sufficient to explain the basic length anisotropies. In particular, is it possible to explain with only these two factors the length asymmetries found in the ‘T’, ‘L’ and ‘+’ configurations, where the latter figure represents a double intersection of the horizontal and vertical segments? Second, are the two factors independent from each other? For instance, across a population of observers, are the observers who are strongly biased by the vertical factor also strongly affected by the bisection factor? Finally, is there a difference in sensitivity between the different configurations? In other words, are observers equally sensitive to discriminate the length of the vertical and horizontal segments in spite of differences in length biases across the different configurations? We address these three questions with a simple model of the vertical–horizontal illusion that indicates that the vertical and bisection biases are not only sufficient but also independent to explain the length biases in the ‘T’, ‘L’, and ‘+’ configurations. In addition, our model predicts a worse length discrimination sensitivity in the ‘+’ configuration as compared to the ‘L’ and our psychophysical results confirm this prediction.

2. Models

We derive here simple models of the vertical–horizontal illusion for various configurations of figures containing a vertical
and a horizontal segment touching at one point (Fig. 1). We distinguish four classes of figures: (1) the ‘L’ where the two segments touch at their extremities, (2) the vertical-‘T’ where the vertical segment bisects the horizontal one, (3) the horizontal-‘T’ where the horizontal segment bisects the vertical one, and (4) the ‘+’-sign where the two segments intersect in their middle.

We present three types of models to describe the comparison of horizontal and vertical lengths for the four classes of figures. None of these models is attempting to explain the phenomenon of the vertical–horizontal illusion. Instead, the models are merely dedicated to summarize as simply and as faithfully as possible the data describing this illusion. The first model is simply assigning a psychometric function per figure class. The next two models are based on the hypothesis that the vertical–horizontal illusion can be reduced to two scaling parameters plus a noise factor. In the second model, this noise is imposed on the image measurements while in the last model, the noise is imposed at the decision stage. We now describe these models in turn.

2.1. Independent psychometric functions

The first model is really used as a benchmark against which we shall compare our two models of interest. We want to model the proportion of times the vertical line appears longer than the horizontal one for various values of the aspect ratio of the figure (ratio of vertical over horizontal lengths), and this for the four stimulus classes. In this first model, we assume that each figure class is characterized by its own specific phenomenon. As such, we assign a psychometric function to each figure class, with its own bias and slope. We shall assume that the shape of the psychometric function is well characterized by a cumulative Normal of the logarithm of the aspect ratio of the stimulus.

Let \( \hat{v} \) (respectively \( \hat{h} \)) represent the perceived length of the vertical (resp. horizontal) segment whose physical length is \( \hat{v} \) (resp. \( \hat{h} \)). For the figure class \( i \) where \( i \in [1, 4] \), the probability that the vertical segment is judged longer than the horizontal one is

\[
p(v > h) = \int_{-\infty}^{\frac{\log(v/h)}{\sigma_i}} \phi(y; \mu_i, \sigma_i) dy,
\]

where \( \phi(x; \mu, \sigma) \) is a short-hand for the Normal distribution with mean \( \mu \), standard deviation \( \sigma \), and evaluated at point \( x \). The mean corresponds to the point of subjective equality, that is the physical length log aspect ratio that leads to a chance probability \( (p = 0.5) \) that the vertical segment is judged longer than the horizontal one. The standard deviation is inversely proportional to the slope of the psychometric function, that the precision with which the observer can report that one segment was longer than the other. Because each psychometric function has two parameters (its mean and its standard deviation), the model as a whole for the four figure classes has eight degrees of freedom. The characteristics of this model are summarized in Table 1.

Fig. 1. Stimuli and illustrations of the late-noise model. Stimuli were grouped into four classes: ‘L’ at four orientations, vertical-‘T’ upright and upside-down, horizontal-‘T’ oriented to the left and right, and the ‘+’-sign. The plots show predictions based on the late-noise model detailed in the text in terms of the probability that the vertical segment is perceived longer than the horizontal one for different aspect ratios. Each plot represents the model predictions for a different pair of anisotropy and bisection parameters \( (a \text{ and } b) \) respectively. For each psychometric curve, two features are important to compare across classes of stimuli and models: the bias represented by the point of subjective equality and the sensitivity represented by the slope of the curve at mid-height. Colours for the psychometric functions represent the classes of stimuli.
Parameters of the three models for the four stimulus classes. The left-most column lists all four classes of stimuli. Then, for each model, the left column reports the point of subjective equality (PSE) for each figure, that is the ratio between the vertical and horizontal lengths for the two of them to appear equally long. The right column reports the standard deviation (SD) of the psychometric function to discriminate the lengths of the vertical and horizontal segments. For the early and late-noise models, the anisotropy parameter \( \gamma \) represents the overestimation of the vertical segment, the bisection parameter \( \beta \) represents the under-estimation of the bisected line, and the constant \( \epsilon \) is proportional to the uncertainty to estimate the length of a segment.

<table>
<thead>
<tr>
<th>Stimulus class</th>
<th>Independent psychometric functions</th>
<th>Early-noise model</th>
<th>Late-noise model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PSE SD</td>
<td>PSE SD</td>
<td>PSE SD</td>
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<tr>
<td>‘L’</td>
<td>( \mu_1 ) ( \sigma_1 )</td>
<td>( \frac{1}{a} ) ( \frac{1}{a} ) ( \frac{1}{a} )</td>
<td>( \frac{1}{a} ) ( \frac{1}{a} ) ( \frac{c}{a} )</td>
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<tr>
<td>Vertical-‘T’</td>
<td>( \mu_2 ) ( \sigma_2 )</td>
<td>( \frac{1}{ab} ) ( \frac{1}{ab} ) ( \frac{1}{ab} )</td>
<td>( \frac{1}{ab} ) ( \frac{1}{ab} ) ( \frac{c}{ab} )</td>
</tr>
<tr>
<td>Horizontal-‘T’</td>
<td>( \mu_3 ) ( \sigma_3 )</td>
<td>( \frac{1}{a} ) ( \frac{1}{a} ) ( \frac{b}{a} )</td>
<td>( \frac{1}{a} ) ( \frac{1}{a} ) ( \frac{bc}{a} )</td>
</tr>
<tr>
<td>‘+’-sign</td>
<td>( \mu_4 ) ( \sigma_4 )</td>
<td>( \frac{1}{a} ) ( \frac{1}{a} ) ( \frac{1}{a} )</td>
<td>( \frac{1}{a} ) ( \frac{1}{a} ) ( \frac{bc}{a} )</td>
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### 2.2. Early-noise model

In the early-noise model, we assume that the measurements of the horizontal and vertical segments of the figure are corrupted by noise. For simplicity, we assume this noise to be normally distributed with zero-mean and fixed variance \( \sigma^2 \). For instance, the measured vertical length \( v \) when the observer is estimating the length of a vertical segment of physical length \( v \) presented on its own is \( v = v + \epsilon \), where \( p(\epsilon) = \phi(\epsilon; 0, \sigma) \) (2) and similarly for the horizontal length. We assume that this variability is the same for all orientations, and that it is also independent of the length of the segment (i.e. Weber’s law is neglected over the range of lengths used in the experiment).

We now introduce the first parameter \( \alpha \) that corresponds to the overestimation of the length of the vertical segment. This parameter will be called the “anisotropy component”. For instance, in the vertical-T configuration, it is the horizontal segment that is bisected, therefore we have

\[
\hat{h} = h/\beta.
\]

(4)

Parameters \( \alpha \) and \( \beta \) are both expected to be larger than one. Taking into account both the overestimation of the vertical line and the under-estimation of the bisected line, the perceived vertical–horizontal length ratio in the vertical-T configuration becomes

\[
\frac{v}{h} = \alpha b \frac{\gamma}{\beta}.
\]

(5)

We see that in the vertical-T configuration, the anisotropy and bisection components combine in a multiplicative way, making this configuration subject to a very strong illusion. At the point of subjective equality (PSE), that is the physical vertical–horizontal length ratio for which the vertical length is perceptually equal to the horizontal length, we have by definition \( \frac{v}{h} = 1 \). Because the measurement noise was unbiased, we can directly extract from Eq. (5) the physical aspect ratio at the PSE for the vertical-T figure, namely

\[
\left( \frac{v}{h} \right)_{PSE} = \frac{\gamma}{\beta} \left( \frac{v}{h} \right)_{PSE} = \frac{1}{ab}.
\]

(6)

Repeating this reasoning for the other three figures, we obtain the PSEs for the four figure classes. These values are collected in Table 1.

In order to compute the precision with which we can discriminate the vertical and horizontal lengths, we need to return to their noise distribution. Combining Eqs. (2) and (3), the distribution of estimated vertical length is

\[
p(v) = \phi(v; \alpha v, \alpha \sigma).
\]

(7)

In the vertical-T configuration, Eq. (4) leads to the expression of the distribution of estimated horizontal lengths, namely

\[
p(h) = \phi(h; h/\beta, \sigma^2/\beta).
\]

(8)

When the lengths of two segments are compared, one sample is taken from each of the two distributions, and the larger value determines the longer percept. The more the two distributions are displaced one relative to the other, the easier it is to discriminate samples from the two distributions, and thus the larger is the probability that one segment is systematically judged longer than the other (see Green & Swets, 1966). In other words, the probability that the vertical segment is judged longer than the horizontal one \( p(v > h) \) increases as the difference between \( \gamma \) and \( \beta \) increases. Therefore, to compute this probability, we have to determine the distribution of the difference between vertical and horizontal estimates. The probability of the difference is normally distributed with a mean equal to the means difference and a variance equal to the sum of the variance of single estimates

\[
p(x = v - h) = \phi \left( x; a v - \frac{h}{\beta}, \sigma \sqrt{a^2 + \frac{1}{b^2}} \right).
\]

(9)

We can now compute the probability that the vertical segment is judged longer than the horizontal one by integrating over the domain where this difference is positive

\[
p(v > h) = \int_{-\infty}^{\infty} \phi \left( x; a v - \frac{h}{\beta}, \sigma \sqrt{a^2 + \frac{1}{b^2}} \right) dx.
\]

(10)

With the change of variable \( y = \frac{x}{a h} + \frac{v}{h} \), this equation can be rewritten to reveal the more usual cumulative Normal function used to fit psychometric functions

\[
p(v > h) = \int_{-\infty}^{1} \phi \left( y; \frac{1}{ab}, \sigma \sqrt{1 + \frac{1}{a b^2}} \right) dy.
\]

(11)

where \( c = \frac{1}{2} \) is a constant (assuming that \( h \) is kept constant; see Section 3 below). This latter equation gives us the point of subjective equality for the ratio of the vertical to horizontal segments so that they are undistinguishable in length (\( \frac{v}{h} \)). We find again the value that we had already computed in Eq. (6). Eq. (11) also gives us the slope of the psychometric function to discriminate these two segments. For a cumulative Normal psychometric function, this slope is inversely proportional to the standard deviation of the
underlying Normal distribution, and in the case of the vertical-‘T’ figure this standard deviation equals \( \sqrt{1 + \frac{1}{\alpha^2}} \).

Repeating this reasoning for the other three figures, we obtain the standard deviations of the psychometric function for the four figure classes. These values are collected in Table 1.

2.3. Late-noise model

The late-noise model is similar to the early-noise model except that the uncertainty on lengths is assumed to play a role near the decision stage rather than directly on the image measurements. In this case, the noise does not undergo the scaling operations that make the vertical line appear longer and the bisected line shorter. Therefore, we can expect some differences in the precision with which vertical and horizontal lengths are compared, although no differences on the biases.

We first start by applying the anisotropy component to the estimation of the vertical length

\[ \hat{v} = a \cdot v. \]

(12)

We then apply the bisection component on the bisected segment. For instance, in the vertical-T configuration, we have

\[ \hat{h} = h / b. \]

(13)

We now introduce some variability on the length estimations. We again assume that this variability is normally distributed with zero-mean and a variance equal to \( \sigma^2 \). For instance, the probability that the particular length \( \hat{h} \) is perceived is

\[ p(\hat{h} = \theta | h, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(\hat{h} - \theta)^2}{2\sigma^2} \right). \]

(14)

where \( \hat{h} \) is the mean estimated length. We then determine the distribution of the difference between vertical and horizontal estimates. The probability of the difference is normally distributed with a mean equal to the means difference and a variance equal to twice the variance of single estimates

\[ p(x = \hat{v} - \hat{h} = \theta | h, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \theta)^2}{2\sigma^2} \right). \]

(15)

We can now compute the probability that the vertical segment is judged longer than the horizontal one

\[ p(\hat{v} > \hat{h}) = \int_{\hat{h}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \hat{h})^2}{2\sigma^2} \right) dx. \]

(16)

For instance, for the vertical-T figure, we have

\[ p(\hat{v} > \hat{h}) = \int_{\theta}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \frac{\theta}{b})^2}{2\sigma^2} \right) dx. \]

(17)

With the same change of variable as before \( y = \frac{x}{\theta} + \frac{\theta}{b} \), this equation can be rewritten as

\[ p(\hat{v} > \hat{h}) = \int_{\frac{\theta}{b}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(y - \frac{\theta}{b})^2}{2\sigma^2} \right) dy, \]

(18)

where \( \theta = \frac{a b}{b} \) is a constant (assuming that \( \theta \) is kept constant; see Section 3). This latter equation gives us the point of subjective equality which is again \( \frac{a}{b} \). Eq. (18) also gives us the slope of the psychometric function to discriminate these two segments from the inverse of the standard deviation of the underlying Normal distribution. In the case of the vertical-T figure this standard deviation equals \( \frac{\theta}{b} \). Therefore, the early and late-noise models predict the same PSEs but different slopes of the psychometric functions.

The different points of subjective equality (PSE) and slopes for all stimulus categories are given in Table 1. To appreciate the relative contribution of the anisotropy and bisection factors of the late-noise model, various combinations of parameters \( \alpha \) and \( \beta \) are shown in Fig. 1.

2.4. Predictions

Several fundamental properties of the vertical–horizontal illusion can be extracted from our early and late-noise models. Referring back to Table 1:

(i) The ‘L’ and ‘+’-sign figures have the same PSE. This common PSE is the inverse of the factor \( \alpha \). For the ‘+’-sign figure, the bijecting effect of the vertical by the horizontal segment is compensated by the bisection of the horizontal by the vertical segment, so the only contribution to the illusion is the orientation anisotropy, just like the ‘L’-figure.

(ii) The vertical-‘T’ and horizontal-‘T’ figures have PSEs that are equally far (in log-units) from the PSE of the ‘L’-figure. Relative to the ‘L’, the PSE of the vertical-‘T’ is shifted to the left by a factor \( \beta \) (in log-units), and the PSE of the horizontal-‘T’ is shifted to the right by the same factor.

(iii) According to the late-noise model, the ‘L’ and ‘+’-sign figures have different sensitivities: the psychometric function for the ‘+’-sign figure has a shallower slope by a factor \( \beta \). The increased difficulty to discriminate the lengths of the ‘+’-sign segments comes from the fact that these segments are overall perceived shorter because they are bisected. The early-noise model predicts that these two figures should have the same sensitivity.

(iv) The vertical-‘T’ and horizontal-‘T’ figures have different sensitivities. According to the late-noise model, the psychometric function of the horizontal-‘T’ figure has a shallower slope by a factor \( \beta \).

We now test these predictions in a psychophysical experiment using all four classes of figures.

3. Methods

3.1. Participants and apparatus

There were 24 participants, 15 women and 9 men (mean age = 41.8 years, standard deviation = 22). Stimuli were shown on a 13” monitor and viewed binocularly from a viewing distance of 57 cm.

3.2. Stimuli

Stimuli consisted of two segments, one horizontal and one vertical, touching at one point (see again Fig. 1). One segment was coloured in blue and the other in red. They were displayed on a uniform white background (luminance of 40 cd/m2). The length of the horizontal line could take one of two values (4.5° or 6° of visual angle). The presentation duration was 1 s.

3.3. Procedure

Participants were instructed to judge the relative length of the orthogonal red and blue segments. In half of the blocks of trials, they were instructed to press a key when the red segment was longer, and hold their response if the blue segment appeared longer (a go/no-go task). In the other half of the blocks, participants were given the opposite instructions. The method of constant stimuli was used to manipulate the aspect ratio of the figure. The aspect ratio was manipulated by maintaining the horizontal segment constant (either 4.5° or 6° of visual angle) and varying the vertical...
length. Eleven aspect ratios were chosen equally spaced on a log-scale between 0.81 and 1.23 (an aspect ratio of 1 represents a vertical length physically identical to the horizontal length). Each of the 11 aspect ratios was presented a total of 32 times for each class of figures and each participant. When a class included several figures at different orientations, the 32 repetitions were divided evenly between the different orientations (so for instance, each of the four ‘L’ shapes was presented 8 times for each aspect ratio). In total, each participant ran 1408 trials broken down into 16 blocks separated by small breaks.

4. Results

The proportion of times the red segment is perceived longer than the blue one is converted into the proportion of times the vertical segment is perceived longer than the horizontal one for each of the nine figures. We first test whether there are any significant differences between the different stimuli of a class (e.g. the four ‘L’-figures). For this purpose, we fit psychometric functions (cumulative Gaussians) to each figure, thereby obtaining a model containing 18 degrees of freedom (one mean and one standard deviation for each of the nine figures). This model is compared to the restricted model where only one psychometric function is used per class (i.e. the model from Section 2.1 referred to independent psychometric functions). Because these two models are nested (the degrees of freedom for the second model are a subset of the ones for the first one), we can use the likelihood-ratio test based on the log-likelihood of the best fits achievable by each model. This test could not reject the hypothesis that the restricted model was as good as the full model ($\chi^2(10) = 8.49, p = 0.097$). Therefore, in the remaining of this paper, the data were pooled across all figures within a class (i.e. all four ‘L’-figures together, both vertical-‘T’ together and both horizontal-‘T’ together). Fig. 2 shows the proportion of times the vertical segment is perceived longer against the aspect ratio of the figure for each of the four classes of figures.

We fitted our three models to the human performance first on the pooled data across observers and then separately for each observer. In both cases, the models agreed well with the data, but there were differences in the goodness of fits. Because the models do not have the same number of parameters, and because one model is not simply nested within another one, we cannot follow the likelihood-ratio test used previously. Instead, we adopt the Akaike Information Criterion (AIC) that takes into account the fact the likelihood-ratio test used previously is not simply nested within another one, we cannot follow the Akaike Information Criterion (AIC) that takes into account the fact the likelihood-ratio test used previously is not simply nested within another one, we cannot follow the likelihood-ratio test used previously.

When the late-noise model was adjusted to the pooled data across the 24 observers, the parameters that led to the best fit were: $a = 1.06, b = 1.16, c = 0.10$. In other words, the anisotropy component had a magnitude of 6% whereas the bisection component reached 16%. Therefore, the vertical–horizontal illusion is a misnomer: the bias in the vertical-‘T’ figure is mostly the result of the bisection of the horizontal line by the vertical line. We then adjusted the late-noise model to each participant and extracted the parameters ‘a’ and ‘b’ from the best fit of the model. In addition, 95% confidence intervals on these estimates were computed by bootstrap. The distribution of these parameters is shown in Fig. 3. Importantly, the two parameters are not correlated (Pearson’s $R = -0.134$) indicating that these two parameters are indeed independent.

The comparison between the goodness of fits of our models has favoured the late-noise model. We now test whether each of the above four predictions (Section 2.4) of our late-noise model is supported by the data. To this purpose, we take advantage of the first model based on independent psychometric functions. Cumulative Gaussians were fitted to each stimulus class for each participant. Each cumulative Gaussian is characterized by two parameters, its mean (PSE) and its standard deviation (inverse sensitivity). We now compare means and standard deviations across stimulus classes.

The first prediction was that the ‘L’ and ‘+’-sign figures should have identical points of subjective equality (PSE). Fig. 4 shows the values of the PSEs for the ‘L’ and ‘+’-sign figures (inverse values of these PSEs are shown because they are both expected to equal parameter ‘a’). Both PSEs are significantly larger than one under a t-test across participants (‘L’: $t(23) = 15.5, p < 0.001$; ‘+’: $t(23) = 12.1, p < 0.001$). In addition, a paired t-test indicated that these PSEs were not significantly different ($t(23) = 1.74, p = 0.091$). The first prediction is therefore satisfied.

The second prediction was that vertical-‘T’ and horizontal-‘T’ figures should have PSEs that are equally spaced (in log-units) from the PSE of the ‘L’-figure. Fig. 4 shows the ratio of PSEs between the ‘L’ and the vertical-‘T’ and between the horizontal-‘T’ and the ‘L’. While both of these differences are close to the ‘b’ parameter as expected, they were significantly different from each other ($t(23) = 3.69, p = 0.001$). The origin of this significant difference appears to be a slightly weaker illusion for the horizontal-‘T’ than predicted by our model. Nevertheless, the bisection present in the horizontal-‘T’ makes this figure appear very different from the ‘L’-figure: the ratio of their PSEs is significantly larger than one ($t(23) = 13.3, p < 0.001$). The second prediction is therefore
The slope of the psychometric functions allowed us to discard an early-noise model. A model where the uncertainty on line length estimation is coming from the image measurements is certainly more conventional than a model where noise is introduced at the decision stage (Pelli, 1991). However, one strong prediction of our early-noise model was that the ‘+’-sign and ‘L’-figures should have the same sensitivity, intuitively because the noise is imposed on lengths that are physically identical for these two figures. Our psychophysical results showed a clear difference in sensitivity between the ‘+’-sign and the ‘L’-figures, and the late-noise model provided a better fit than the early-noise model.

The bisection parameter was a shortening of the perceived length of the bisected line rather than the opposite. According to our model, the bisection parameter is responsible for the worse sensitivity for the ‘+’-sign figure as compared to the ‘L’-figure. If the bisection parameter had the opposite effect (i.e., increasing the perceived length of the bisecting line), then the ‘+’-sign figure would lead to a steeper psychometric function. Our third prediction specifically addressed this issue and because it was satisfied, we can assert that a line bisected by another of roughly equal length appears shorter than it would if it were not cut.

Our purpose was to present a minimal model that could account for the main effects present in the vertical–horizontal illusion, both in terms of bias and sensitivity. We managed to achieve our goal with a simple model that includes only three parameters. Obviously, one can extend our model to account more precisely for our data, in particular with respect to our second prediction that was only partially satisfied. Future extensions of the present model can also include the effects of intersecting a segment not necessarily in its middle (Charras & Lupiáñez, 2010) and intersecting two segments not necessarily at right angles (Wolfe et al., 2005). In the meantime, we expect that the model presented here will be useful to investigate independently the horizontal–vertical anisotropy and the bisected-line length estimation.

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References


